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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS.

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TECHNICAL MEMORANDUM 110

THE STRESSES IN COLUMNS UNDER COMBINED AXIAL AND SIDE LOADS.

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# THE STRESSES IN COLUMNS UNDER COMBINED AXIAL AND SIDE LOADS.

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The problem before us is to determine the total stresses in an axially loaded column of any degree of restraint which is also subject to transverse bending both from a uniformly distributed load and from concentrated loads. The solution of this problem is of especial importance in the design of aircraft which consist of slender columns.

Consider the general case (Fig. 1).

A column of length  $L$  which is encastre'd at the points of support by the moments  $M_a$  and  $M_b$  and supports the axial load  $P$  is also subjected to bending by both a uniformly distributed load of  $w$  and concentrated loads.

The length of the column is divided into  $(n + 1)$  sections by the  $n$  concentrated loads.

If  $M_0$  is the simple bending moment at a section between  $a_r$  and  $a_{r+1}$  and  $M_r$  is the total bending moment then

$$M_r = M_0 + Py_r$$

The differential equation of the elastic curve is

$$EI \frac{d^2y}{dx^2} = -M_r$$

By introducing  $c = \sqrt{\frac{EI}{P}}$  we can write

$$c^2 \frac{d^2y}{dx^2} + y = -\frac{M_0}{P}$$

For each one of the  $(n+1)$  sections there is a differential equation of the above form from which

$$M_0 = \frac{1}{2} wLx - \frac{1}{2} wx^2 + \sum_{r=1}^{r=n} F_r \frac{L-a_r}{L} x - \sum_{r=r+1}^{r=n} F_r (x-a_r) + M_a + \frac{M_b - M_a}{L} x$$

Between  $a_r$  and  $a_{r+1}$  the equation of the deflection curve is  $y_r = a_r \cos \frac{x}{c} + B_r \sin \frac{x}{c} - \frac{1}{P} (w c^2 + M_0)$  and the derivative for the tangent to this curve is

$$y'_r = -\frac{1}{c} a_r \sin \frac{x}{c} + \frac{1}{c} B_r \cos \frac{x}{c} - \frac{M'_0}{P}$$

in which

$$M'_0 = \frac{1}{2} wL - wx + \sum_{r=1}^{r=n} F_r \frac{L - a_r}{L} - \sum_{r=r+1}^{r=n} F_r + \frac{M_b - M_a}{L}$$

For the determination of the  $2(n+1)$  constants A and B the following conditions are used:

$$\begin{aligned} \text{Once for } x = 0 & \quad \text{-----} \quad y_n = 0 \\ \text{Once for } x = L & \quad \text{-----} \quad y_0 = 0 \\ n \text{ times for } x = a_r & \quad \text{-----} \quad y_{r-1} = y_r \\ n \text{ times for } x = a_r & \quad \text{-----} \quad y'_{r-1} = y'_r \end{aligned}$$

The solution of these  $(2n + 2)$  equations gives the following values for the constants:

$$A_r = A_n + c \sum_{r=r+1}^{r=n} \frac{F_r}{P} \sin \frac{a_r}{c}$$

wherein

$$A_n = \frac{1}{P} (wc^2 + M_a)$$

$$B_r = B_0 + c \sum_{r=1}^{r=r} \frac{F_r}{P} \cos \frac{a_r}{c}$$

in which

$$B_0 = \frac{1}{P} (wc^2 + M_b) \frac{1}{\sin \frac{L}{c}} - \frac{1}{P} (wc^2 + M_a) \cot \frac{L}{c} - c \cot \frac{L}{c} \sum_{r=1}^{r=n} \frac{F_r}{P} \sin \frac{a_r}{c}$$

By the help of these constants the form of the shear and moment curves can be determined, between  $a_{r+1}$  and  $a_r$ .

$$\text{The moment } M_r = P (A_r \cos \frac{x}{c} + B_r \sin \frac{x}{c}) - wc^2$$

$$\text{The shear } V_r = \frac{P}{c} (-A_r \sin \frac{x}{c} + B_r \cos \frac{x}{c})$$

The maximum bending moment occurs at the abscissa  $x_m$  if  $\tan \frac{x_m}{c} = \frac{B_r}{A_r}$  gives a real value of  $x_m$  falling between  $a_{r+1}$  and  $a_r$ .

Knowing the moments the deflection is then determined from

$$y_r = \frac{1}{P} (M_r - M_0)$$

1. Special case with a numerical example (Figs. 2 and 3).

A freely supported beam is subjected to only one transverse load. The constants are then

$$A_1 = 0$$

$$A_0 = c \cdot \frac{F_1}{P} \sin \frac{a_1}{c}$$

$$B_0 = -c \cdot \frac{F_1}{P} \sin \frac{a_1}{c} \cot \frac{L}{c}$$

$$B_1 = B_0 + c \cdot \frac{F_1}{P} \cos \frac{a_1}{c}$$

The moment between

$$x = 0 \text{ to } a_1 \text{ is } M_1 = PB_1 \sin \frac{x}{c}$$

and

$$x = a_1 \text{ to } L \text{ is } M_0 = P(A_0 \cos \frac{x}{c} + B_0 \sin \frac{x}{c})$$

The maximum moment occurs at the point  $x_m$  as long as

$\tan \frac{x_m}{c} = -\cot \frac{L}{c}$  gives a value  $x_m = L - \frac{3}{2}\pi$  which is greater than  $a_1$ .

That is: If the load is applied within the portion from

$x = 0$  to  $x = L - \frac{c}{2}\pi$  the maximum bending moment always comes at the point  $x = L - \frac{c}{2}\pi$  and not under the load.

A beam of  $L = 4.0$  meters span and with

$$c = \sqrt{\frac{EI}{P}} = 2.0 \text{ meters carries at a distance of}$$

$a_1 = 0.5$  meters from the left support a load  $F_1 = 250$  kg.

At the same time it is subjected to an axial load of  $P = 5000$  kg.

From the quantities

$$\frac{cF_1}{P} = \frac{1}{10} ; \sin \frac{a_1}{c} = 0.248 ; \cos \frac{a_1}{c} = 0.969 ;$$

$$\cot \frac{L}{c} = - 0.458$$

One obtains the values of the constants

$$A_0 = 0.0248$$

$$B_0 = 0.0114$$

$$B_1 = 0.1083$$

The moments

$$M_1 = 545.5 \sin \frac{x}{c}$$

$$M_0 = 124 \cos \frac{x}{c} + 57 \sin \frac{x}{c}$$

The point of maximum moment  $x_m = 0.86$  m.

In Fig. 3 the moment curves are shown. The point of maximum moment does not lie under the load but at section  $x_m = 0.86$  m.

2. Special Case. The loads are located symmetrically about the middle of the beam. (Fig. 4). From the above we find the constants B

$$B_0 = \frac{1}{P} (wc^2 + Ma) \tan \frac{L}{2c} + \frac{c}{2} \sum_{r=1}^{r=n} \frac{F_r}{P} \sin \frac{a_r}{c}$$

$$B_r = B_0 + c \sum_{r=1}^{r=r} \frac{F_r}{P} \cos \frac{a_r}{c}$$

Since the constants  $A$  remain unchanged the maximum bending moment in the middle of the beam is

$$M = \frac{1}{\cos \frac{L}{2c}} \left[ wc^2 \left( 1 - \cos \frac{L}{2c} \right) + \frac{c}{2} F_0 \sin \frac{L}{2c} + c \sum_{r=1}^{r=n} F_r \sin \frac{a_r}{c} + M_a \right]$$

From this formula all the individual cases can be derived.

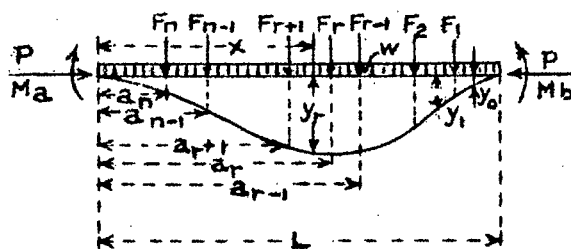


Fig. 1.

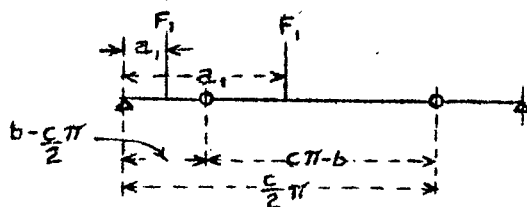


Fig. 2.

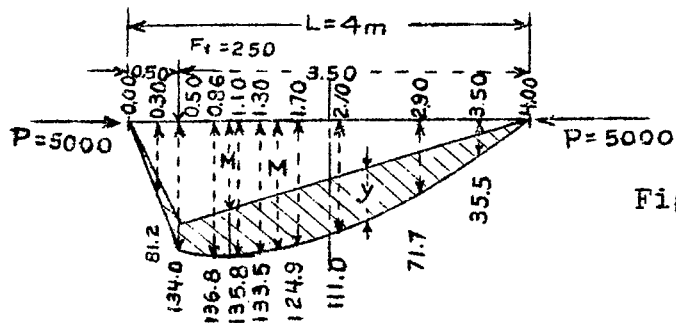


Fig. 3.

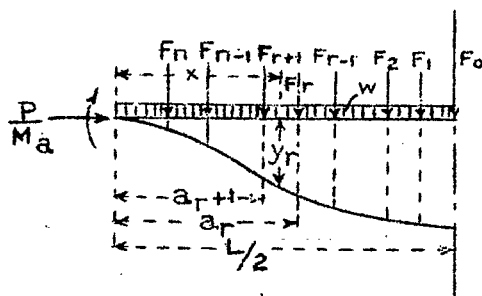


Fig. 4.